Week 3 - Friday

COMP 2230

Last time

- Recurrence relations
- Solving recurrence by iteration

Questions?

Assignment 2

Back to Solving Recurrence Relations by Iteration

Geometric sequence

- Find a pattern for the following recurrence relation:
  - $a_k = ra_{k-1}, k \geq 1$
  - $a_0 = a$
- Again, start at the first term
- Write the next below
- Do not combine like terms!
- Leave everything in expanded form until patterns emerge

Geometric sequence

- It appears that any geometric sequence with the following form
  - $a_k = ra_{k-1}, k \geq 1$
- is equivalent to
  - $a_n = a_0 r^n$, for all integers $n \geq 0$
- This result applies directly to compound interest calculation
- Let's prove it

Employing outside formulas

- Sure, intelligent pattern matching gets you a long way
- However, it is sometimes necessary to substitute in some known formula to simplify a series of terms
- Recall
  - Geometric series: $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$
  - Arithmetic series: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

How many edges are in a complete graph?

- In a complete graph, every node is connected to every other node
- If we want to make a complete graph with k nodes, we can take a complete graph with $k - 1$ nodes, add a new node, and add $k - 1$ edges (so that all the old nodes are connected to the new node
- Recursively, this means that the number of edges in a complete graph is
  - $s_k = s_{k-1} + (k - 1), k \geq 2$
  - $s_1 = 0$ (no edges in a graph with a single node)
- Use iteration to solve this recurrence relation

Mistakes were made!

- You might make a mistake when you are solving by iteration
- Consider the following recursive definition
  - $c_k = 2c_{k-1} + k, k \geq 1$
  - $c_0 = 1$
- Careless analysis might lead you to the explicit formula
  - $c_n = 2^n + n, n \geq 0$
- How would this be caught in a proof by induction verification?

Solving Second-Order Linear Homogeneous Relations with Constant Coefficients

Second-order linear homogeneous relations with constant coefficients

- Second-order linear homogeneous relations with constant coefficients are recurrence relations of the following form:
  - $a_k = Aa_{k-1} + Ba_{k-2}$ where $A, B \in \mathbb{R}$ and $B \neq 0$
- These relations are:
  - **Second order** because they depend on a_{k-1} and a_{k-2}
  - **Linear** because a_{k-1} and a_{k-2} are to the first power and not multiplied by each other
  - **Homogeneous** because there is no constant term
  - **Constant coefficients** because A and B are fixed values

Why do we care?

- I'm sure you're thinking that this is an awfully narrow class of recurrence relations to have special rules for
- It's true: There are many (infinitely many) ways to formulate a recurrence relation
  - Some have explicit formulas
  - Some do not have closed explicit formulas
- We care about this one partly for two reasons
  1. We can solve it
  2. It lets us get an explicit formula for Fibonacci!

Pick 'em out

- Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

a) $a_k = 3a_{k-1} + 2a_{k-2}$

b) $b_k = b_{k-1} + b_{k-2} + b_{k-3}$

c) $c_k = \frac{1}{2}c_{k-1} - \frac{3}{7}c_{k-2}$

d) $d_k = d_{k-1}^2 + d_{k-1}d_{k-2}$

e) $e_k = 2e_{k-2}$

f) $f_k = 2f_{k-1} + 1$

g) $g_k = g_{k-1} + g_{k-2}$

h) $h_k = (-1)h_{k-1} + (k-1)h_{k-2}$

Characteristic equation

- We will use a tool to solve a SOLHRRwCC called its **characteristic equation**
- The characteristic equation of $a_k = Aa_{k-1} + Ba_{k-2}$ is:
  - $t^2 - At - B = 0$ where $t \geq 0$
- Note that the sequence $1, t, t^2, t^3, \ldots, t^n$ satisfies $a_k = Aa_{k-1} + Ba_{k-2}$ if and only if t satisfies $t^2 - At - B = 0$

Demonstrating the characteristic equation

- We can see that $1, t, t^2, t^3, \ldots, t^n$ satisfies $a_k = Aa_{k-1} + Ba_{k-2}$ if and only if t satisfies $t^2 - At - B = 0$ by substituting in t terms for a_k as follows:
  - $t^k = At^{k-1} + Bt^{k-2}$

- Since $t \geq 0$, we can divide both sides through by t^{k-2}
  - $t^2 = At + B$
  - $t^2 - At - B = 0$

Using the characteristic equation

- Consider $a_k = a_{k-1} + 2a_{k-2}$
- What is its characteristic equation?
  - $t^2 - t - 2 = 0$
- What are its roots?
  - $t = 2$ and $t = -1$
- What are the sequences defined by each value of t?
  - $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$
  - $1, -1, 1, -1, \ldots, (-1)^n, \ldots$
- Do these sequences satisfy the recurrence relation?

Finding other sequences

- An infinite number of sequences satisfy the recurrence relation, depending on the initial conditions
- If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same SOLHRRwCC, then, for any $C, D \in \mathbb{R}$, the following sequence also satisfies the SOLHRRwCC
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$

Finding a sequence with specific initial conditions

- Solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2}$ where $a_0 = 1$ and $a_1 = 8$
- The result from the previous slide says that, for any sequence r_k and s_k that satisfy a_k
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$
- We have these sequences that satisfy a_k
  - $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$
  - $1, -1, 1, -1, \ldots, (-1)^n, \ldots$
- Thus, we have
  - $a_0 = 1 = C2^0 + D(-1)^0 = C + D$
  - $a_1 = 8 = C2^1 + D(-1)^1 = 2C - D$
- Solving for C and D, we get:
  - $C = 3$
  - $D = -2$
- Thus, our final result is $a_n = 3 \cdot 2^n - 2(-1)^n$

Distinct Roots Theorem

- We can generalize this result
- Let sequence $a_0, a_1, a_2, \ldots$ satisfy:
  - $a_k = Aa_{k-1} + Ba_{k-2}$ for integers $k \geq 2$
- If the characteristic equation $t^2 - At - B = 0$ has two distinct roots r and s, then the sequence satisfies the explicit formula:
  - $a_n = Cr^n + Ds^n$
  - where C and D are determined by a_0 and a_1

Now we can solve Fibonacci!

- Fibonacci is defined as follows:
  - $F_k = F_{k-1} + F_{k-2}, k \geq 2$
  - $F_0 = 1$
  - $F_1 = 1$
- What is its characteristic equation?
  - $t^2 - t - 1 = 0$
- What are its roots?
- $t = \frac{1 \pm \sqrt{5}}{2}$
- Thus, $F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$

We just need C and D

- So, we plug in $F_0 = 1$ and $F_1 = 1$
- $F_0 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1$
- $F_1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right)$
- Solving for C and D, we get
- $C = \frac{1+\sqrt{5}}{2\sqrt{5}}$
- $D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$
- Substituting in, this yields
- $F_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-(1-\sqrt{5})}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

Single-Root Case

- Our previous technique only works when the characteristic equation has distinct roots r and s
- Consider sequence $a_k = Aa_{k-1} + Ba_{k-2}$
- If $t^2 - At - B = 0$ only has a single root r, then the following sequences both satisfy a_k
  - $1, r^1, r^2, r^3, \ldots r^n, \ldots$
  - $0, r, 2r^2, 3r^3, \ldots nr^n, \ldots$

Single root equation

- Our old rule said that if $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same SOLHRRwCC, then, for any $C, D \in \mathbb{R}$, the following sequence also satisfies the SOLHRRwCC
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$
- For the single root case, this means that the explicit formula is:
  - $a_n = Cr^n + Dnr^n$, for integers $n \geq 0$
  - where C and D are determined by a_0 and a_1

SOLHRRwCC TL;DR

- To solve sequence $a_k = Aa_{k-1} + Ba_{k-2}$
- Find its characteristic equation $t^2 - At - B = 0$
- If the equation has two distinct roots r and s
  - Substitute a_0 and a_1 into $a_n = Cr^n + Ds^n$ to find C and D
- If the equation has a single root r
  - Substitute a_0 and a_1 into $a_n = Cr^n + Dnr^n$ to find C and D

Upcoming

Next time…

- General recursive definitions

Reminders

- Work on Assignment 2
  - Due next Friday
- **No class on Monday!**
- Read 5.9 for Wednesday